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SYMMETRY CRITERIA FOR THE EQUALITY OF INTERIOR AND EXTERIOR HEAT CONDUCTION SHAPE FACTORS WITH EXACT SOLUTIONS

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1. ABSTRACT

In several geometries, the two-dimensional heat conduction shape factor of the interior of a simply connected region (Ω) is exactly equal to that of its exterior ($\mathbb{R}\setminus\Omega$) under the same boundary conditions. Recent work has conjectured that this equality must always hold. We present a counterexample to that conjecture and provide a sufficient condition for interior-exterior shape factor equality by exploiting a beautiful and little-known symmetry method due to Hersch [1] which we introduce in a tutorial manner. We also provide an exact solution for all bodies that have the required symmetry.

2. INTRODUCTION

Consider a closed bounded boundary curve ∂B in the plane, where sections of the curve are held at one of two temperatures, T_0 and $T_0 + \Delta T$, with the segments at different temperatures isolated by adiabatic segments. Then the conductive heat transfer Q, between sections at different temperatures, can be written as $Q = kS\Delta T$, where k is the thermal conductivity of the medium and S the shape factor, a geometrically determined constant. Notably S is invariant under conformal transformations of the domain [2].

To determine S, one must solve a Laplace problem for the temperature field $T(x, y)$ and then S can be computed by integrating the heat flux out of the high temperature boundary. For the same boundary conditions, there is no immediately obvious reason why S should take on the same value when considering the heat transfer to take place on the interior of ∂B and the exterior of ∂B. Nonetheless, it has been observed for many problems that $S_{int} = S_{ext}$ [2]. It was then conjectured that $S_{int} = S_{ext}$ must always hold. However, our FEM computations demonstrate a counterexample: equality does not hold for arbitrary rectangles [3].

In the present paper we show that the equality of interior and exterior shape factors observed in the specific examples of [2] stems from a conformal equivalence between the interior and exterior problems. We then apply the Schwarz reflection principle to derive a sufficient set of criteria which guarantee $S_{int} = S_{ext}$.

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The criteria comprise a symmetry condition on both the shape of ∂B as well as a condition on the boundary condition placement along ∂B . Notably, the examples from [2] constitute satisfy our criteria.

3. METHDOLOGY

Two shape factor problems are conformally equivalent if there is a conformal map between the two domains which maps boundary conditions in the pre-image boundary to the same boundary condition at the image location. While the Riemann mapping theorem guarantees that there always exists a conformal mapping between the interior and exterior of a simple closed curve, only three points on the boundary can be assigned. The heat conduction problem requires that four points be assigned. Thus, the interior and exterior problems with the same boundary conditions on ∂B are not necessarily conformally equivalent: in general, their shape factors may differ

We exploit the Schwarz reflection principle to derive a class of geometries ∂B , with accompanying boundary conditions, whose interior and exterior *do* possess a conformal equivalence. The Schwarz reflection principle states that if a function $W(z)$ in the complex plane is analytic in some domain $D \in \mathbb{C}$ with $\mathfrak{R}\{D\} \leq 0$, and satisfies $W(\mathbb{R}) \to \mathbb{R}$, then W may be extended to be analytic over the reflected domain $D \cup \overline{D}$ such that $W(\overline{z}) = \overline{W(z)}$, where the overline denotes complex conjugation. Since a conformal map is simply a single-valued analytic function defined over a domain, the Schwarz reflection principle can be then used to construct conformal maps.

By exploiting the Schwarz reflection principle, we build interior-to-exterior conformal maps, that preserve portions of the boundary, provided that the domain can be constructed from sector reflections of primitive shapes. Some examples are given in Fig. 1; each shape therein can be constructed from repeated sector reflections of a primitive shape (shaded in purple). Our analysis reveals the existence of an interior-to-exterior conformal map associated with each domain in Fig. 1, which maps each primitive edge segment to itself. Thus, if boundary conditions are specified as unchanging along each primitive edge, then the interior and exterior problems are conformally equivalent and thus possess the same shape factor. Moreover, by consideration of the cross-ratio and mapping from the unit disc, we derive exact solutions for all geometries obeying our symmetry criteria.

4. RESULTS

Fig. 1 illustrates a variety of shapes that may be constructed from sector reflections of a primitive shape. As follows from the previous section, when boundary conditions are unchanging along each primitive edge, the interior and exterior shape factors must be equal. Our exact solution, in terms of an elliptic function, applies to all geometries satisfying our symmetry criteria, including all shapes in Fig. 1.

A square can be constructed by repeatedly reflecting a triangle constituting 1/8 of the square. Thus, the primitive edge along which boundary conditions must be set is a half-face of the square. More generally, the boundary conditions may be set along half-faces of regular polygons: a hexagon is analysed in Fig. 2.

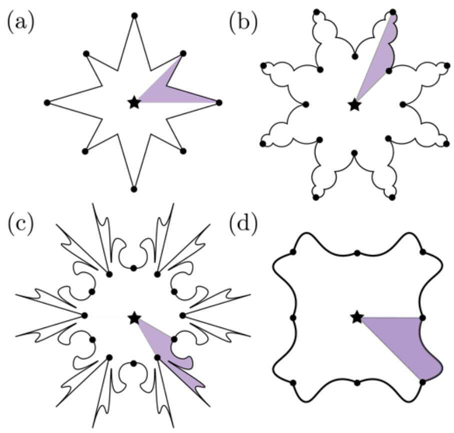

Fig. 1 Examples of shapes with the necessary symmetry. When boundary conditions are unchanging along each primitive boundary segment, $S_{int} = S_{ext}$.

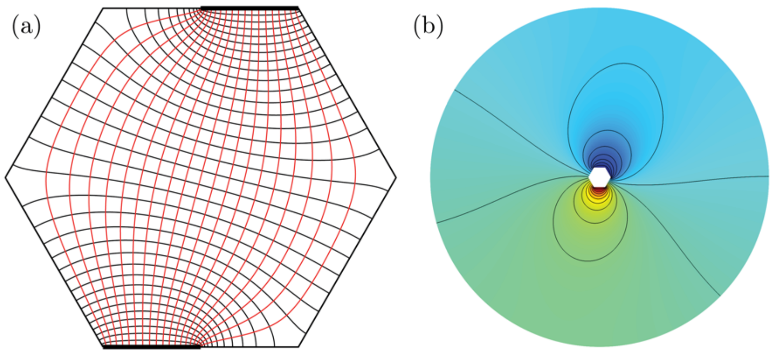

Fig. 2 FEM simulations of an interior (a) and exterior (b). The shape factors are computed with numerical integration and found to be $S_{int} = 0.5776$ and $S_{ext} = 0.5700$, in good agreement. The numerical results are also in agreement with the exact solution for the interior problem, $S_{exact} = 1/\sqrt{3} \approx 0.57735\cdots$, as given by Hersch [1].

5. CONCLUSIONS

We have shown that interior and exterior shape factors need not always be equal. We have used the Schwarz reflection principle to derive a class of geometries possessing equal interior and exterior shape factors. Known historic examples with $S_{int} = S_{ext}$ are found to be special cases of our general class of geometries. More details can be found in our recently published paper [3].

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