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RECENT PROGRESSES ON FUNDAMENTALS AND APPLICATIONS OF COMPUTATIONAL-ANALYTICAL INTEGRAL TRANSFORMS

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1. ABSTRACT

The present lecture reviews the analytic-based methodology known as the Generalized Integral Transform Technique (GITT) for convection-diffusion problems, focusing on recent progresses on fundamentals, such as the single domain formulation and the nonlinear eigenvalue problem base, which are more closely reviewed. Also, its recent application in direct-inverse analysis in petroleum reservoir simulation is illustrated.

2. INTRODUCTION

The integral transform method is a well-known analytical methodology for the exact solution of linear partial differential equations in mathematical physics, with roots in the separation of variables approach proposed by Fourier in the first half of the 19th century, and in the work of Koshlyakov in the first half of the 20th century, followed by the extensions advanced by Luikov, Olçer, Ozisik, and Mikhailov, among others. Despite its wide use and evolution, the classical approach does not apply to several non-transformable problems, such as most of the nonlinear formulations in transport phenomena. For this reason, this approach was progressively generalized to a hybrid computational-analytical structure, known as the Generalized Integral Transform Technique (GITT) [1,2], which allows for a flexibilization in the numerical solution of the resulting coupled transformed ordinary differential systems. The GITT has been advanced to handle different classes of problems in heat and fluid flow, previously only solvable by purely discrete approaches, offering advantages in terms of accuracy, robustness, and computational effort at the price of further analytical involvement. It has been extensively adopted as a benchmarking tool but also as a computational tool in itself in CPU-intensive tasks, such as inverse problem analysis, optimization, and stochastic simulations. This lecture summarizes the formalism in the consolidated GITT approach and briefly discusses more recent progresses that extend its applicability. A single-domain reformulation strategy was proposed in [3], originally aimed at solving conjugated heat transfer problems when solid and fluid subregions would require separate integral transformations or solving coupled eigenvalue problems. The strategy is based on rewriting the convection-diffusion equations for each subdomain, with their respective physical properties and source terms, as one single formulation for the whole region, with spatially variable coefficients that vary abruptly at the interfaces of the subregions, recovering the heterogeneities. A new formalism was proposed [4] based on a nonlinear eigenvalue problem choice, carrying along to the eigenfunction expansion based on the nonlinear behavior of the original problem coefficients and operators. This more general solution path provides a formal solution

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that encompasses traditional formalism with a linear eigenvalue problem, as shown below and leads to improved convergence rates. Recent applications have focused on CPU-intensive problems, such as in inverse problem analysis, when numerous computational runs of the direct problem are, in general, required. The merits of the hybrid approach are then more evident in both precision and computational speed. The general formalism in the GITT is next presented, followed by an illustration of its application in direct-inverse problem analysis.

3. PROBLEM FORMULATION AND FORMAL SOLUTION

Consider a general convection-diffusion problem for M coupled potentials, $T_k(\mathbf{x},t)$, k=1,2,...,M, with a nonlinear velocity vector $\mathbf{u}(\mathbf{x},t,\mathbf{T})$ and nonlinear source terms $P_k(\mathbf{x},t,\mathbf{T})$ and $\phi_k(\mathbf{x},t,\mathbf{T})$:

$$w_k(\mathbf{x},t,\mathbf{T})\frac{\partial T_k(\mathbf{x},t)}{\partial t}+\mathbf{u}(\mathbf{x},t,\mathbf{T}).\nabla T_k=\nabla.K_k(\mathbf{x},t,\mathbf{T})\nabla T_k-d_k(\mathbf{x},t,\mathbf{T})T_k+P_k(\mathbf{x},t,\mathbf{T}),\ \text{in}\ \mathbf{x}\in V,\ t>0,\ k=1,2,...,M \quad (1)$$

with initial (or inlet) and boundary conditions given by

$$T_k(\mathbf{x},0)=f_k(\mathbf{x})\ ,\ \mathbf{x}\in V \qquad \alpha_k(\mathbf{x},t,\mathbf{T})T_k(\mathbf{x},t)+\beta_k(\mathbf{x},t,\mathbf{T})K_k(\mathbf{x},t,T_k)\frac{\partial T_k}{\partial \mathbf{n}}=\phi_k(\mathbf{x},t,\mathbf{T}),\ \mathbf{x}\in S \quad (2,3)$$

Problem (1-3) can be rewritten with characteristic linear coefficients that have only $\mathbf{x}$ dependence, i.e., $w^*(\mathbf{x})$, $K^*(\mathbf{x})$, $d^*(\mathbf{x})$, $\alpha^*(\mathbf{x})$ and $\beta^*(\mathbf{x})$, while the modified nonlinear source terms then incorporate the remaining nonlinear portions of the equation and boundary conditions operators, including the nonlinear convection term:

$$w_k^*(\mathbf{x})\frac{\partial T_k(\mathbf{x},t)}{\partial t}=\nabla.K_k^*(\mathbf{x})\nabla T_k-d_k^*(\mathbf{x})T_k+P_k^*(\mathbf{x},t,\mathbf{T})\ ,\ \text{in}\ \mathbf{x}\in V,\ t>0,\ k=1,2,...,M \quad (4)$$

with initial/inlet and boundary conditions given by

$$T_k(\mathbf{x},0)=f_k(\mathbf{x}),\ \mathbf{x}\in V \qquad \alpha_k^*(\mathbf{x})\ T_k(\mathbf{x},t)+\beta_k^*(\mathbf{x})\ K_k^*(\mathbf{x})\ \frac{\partial T_k}{\partial \mathbf{n}}=\phi_k^*(\mathbf{x},t,\mathbf{T})\ ,\quad \mathbf{x}\in S\ ,\ t>0 \quad (5,6)$$

The characteristic linear coefficients in Eqs. (4-6) lead to the preferred decoupled linear eigenvalue problems, obtained from the separation of variables applied to the homogeneous version of the problem (4-6), as:

$$\nabla.K_k^*(\mathbf{x})\nabla\psi_{k,i}(\mathbf{x})+\left[\mu_{k,i}^2 w_k^*(\mathbf{x})-d_k^*(\mathbf{x})\right]\psi_{k,i}(\mathbf{x})=0\ ,\ \mathbf{x}\in V \quad (7)$$

$$\alpha_k^*(\mathbf{x})\psi_{k,i}(\mathbf{x})+\beta_k^*(\mathbf{x})K_k^*(\mathbf{x})\frac{\partial\psi_{k,i}(\mathbf{x})}{\partial\mathbf{n}}=0,\ \mathbf{x}\in S \quad (8)$$

Then, the following integral transform pair with the normalization integrals is defined:

$$\bar{T}_{k,i}(t)=\int_V w_k^*(\mathbf{x})\psi_{k,i}(\mathbf{x})T_k(\mathbf{x},t)\,dv\ ,\quad \text{transform} \quad (9)$$

$$T_k(\mathbf{x},t)=\sum_{i=1}^{\infty}\frac{1}{N_{k,i}}\psi_{k,i}(\mathbf{x})\bar{T}_{k,i}(t)\ ,\quad \text{inverse} \quad N_{k,i}=\int_V w_k^*(\mathbf{x})\psi_{k,i}^2(\mathbf{x})\,dv,\ \text{norms} \quad (10,11)$$

Integral transforming Eq. (4) through the operator $\int_V(-)\psi_{k,i}(\mathbf{x})dv$, leads to the ODE system for the transformed potentials, $\bar{T}_{k,i}(t)$, and corresponding transformed initial conditions, written as:

$$\frac{d\bar{T}_{k,i}(t)}{dt}+\mu_{k,i}^2\bar{T}_{k,i}(t)=\bar{g}_{k,i}(t,\bar{\mathbf{T}}),\ 0<t<t_f,\ k=1,2,...,M,\ i=1,2,... \quad (12)$$

$$\bar{T}_{k,i}(0)=\bar{f}_{k,i}\ ,\ \text{where}\ \ \bar{f}_{k,i}=\int_V w_k^*(\mathbf{x})\ \psi_{k,i}(\mathbf{x})\ f_k(\mathbf{x})\ dv \quad (13,14)$$

$$\bar{g}_{k,i}(t,\bar{\mathbf{T}})=\int_V\psi_{k,i}(\mathbf{x})P_k^*(\mathbf{x},t,\mathbf{T})dv+\int_S\phi_k^*(\mathbf{x},t,\mathbf{T})\left[\frac{\psi_{k,i}(\mathbf{x})-K_k^*(\mathbf{x})\dfrac{\partial\psi_{k,i}}{\partial\mathbf{n}}}{\alpha_k^*(\mathbf{x})+\beta_k^*(\mathbf{x})}\right]ds \quad (15)$$

4. APPLICATION

Core flood experiments are critical for understanding the dynamics of multiphase flows in heterogeneous porous media and allow researchers to simulate and study the effectiveness of various oil recovery techniques from petroleum reservoirs. These experiments provide important data for inverse problems, enabling the determination of intrinsic properties of the plug subjected to fluid flow, such as relative permeabilities. The physical problem involves a heterogeneous plug saturated with oil which undergoes axial water injection at one end, producing oil and water (after breakthrough) at the opposite end. Selected core samples from actual reservoirs (Fig.1) have been studied through microtomography and their porosity was properly mapped (Fig.2). The direct problem solution is obtained via GITT, for both one- and three-dimensional heterogeneous media with capillary pressure effects, and its convergence is illustrated in Fig.3, for N=30, $N^{*}<25$. For the inverse problem, the Monte Carlo via Markov Chain (MCMC) method was employed, using synthetic experimental data, and the estimated water and oil relative permeabilities are shown in Fig.4, against the exact functions.

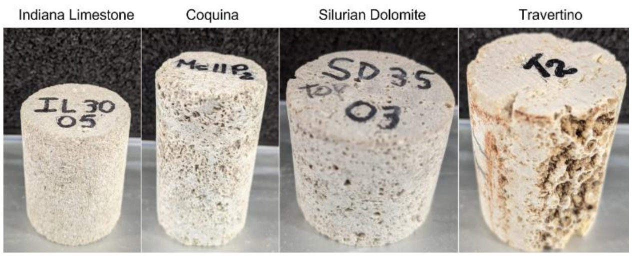

Fig.1 – Selected reservoir plug samples considered

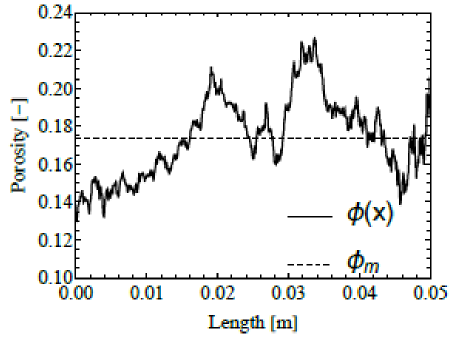

Fig.2 – Typical variable porosity: microtomography

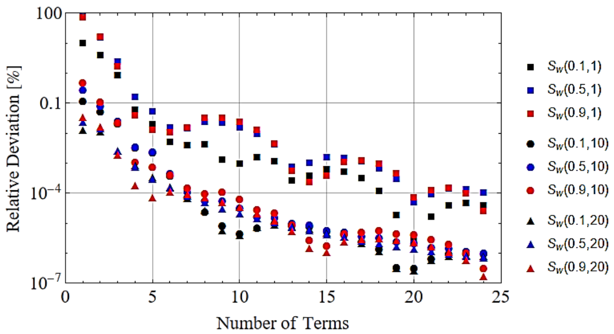

Fig.3 – Convergence of GITT for water saturation

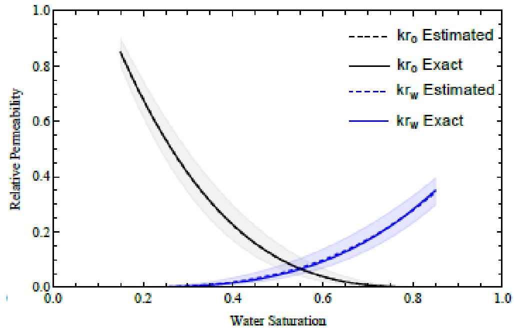

Fig.4 – Estimated and exact relative permeabilities

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